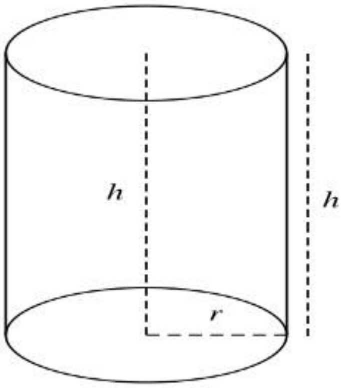
Surfaces areas of revolutions

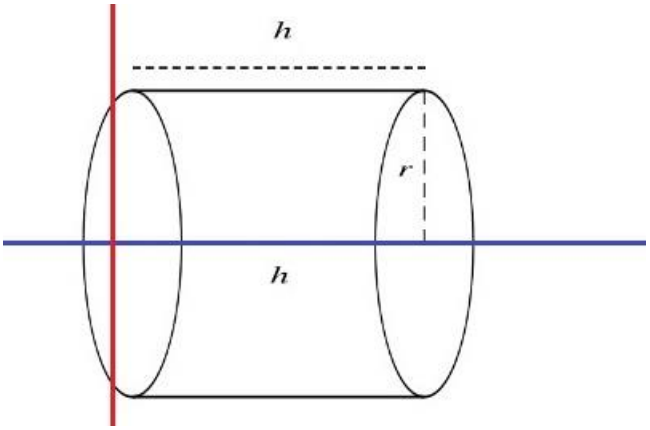
This guide will show you how to set up integrals to solve for surface areas of a curve rotated about an axis. To get through as many possible scenarios as I can, I wont be solving the integrals as they are fairly simple. I believe most people have trouble with the setup, rather than the actual integration.



Surface Area = 2 * PI * R * H

Let’s first start with something we already know, the surface area of a cylinder. Ignoring the end caps.
Our equation would be the circumfrance (2 * PI * R) multiplied by the height of the cylinder (H)

Now visualize that same cylinder placed on our X (Blue) and Y (Red) axis. Obviously, our surface area remains unchanged. So if we know the length (height) of the side of a cylinder, and multiply it by the radius, we can figure out the surface area. Keep this concept in mind, we can combine it with other concepts we already know to easily solve for the surface area of a function revolved around an axis.



For any given function, we can use our arc length formula to figure out our “height” value. We can also use the given function to figure out our radius. Putting it all together we arrive at our formula for surface of revolution around the X axis. And our limits of integration (A and B) are our start and end points on the X-axis.

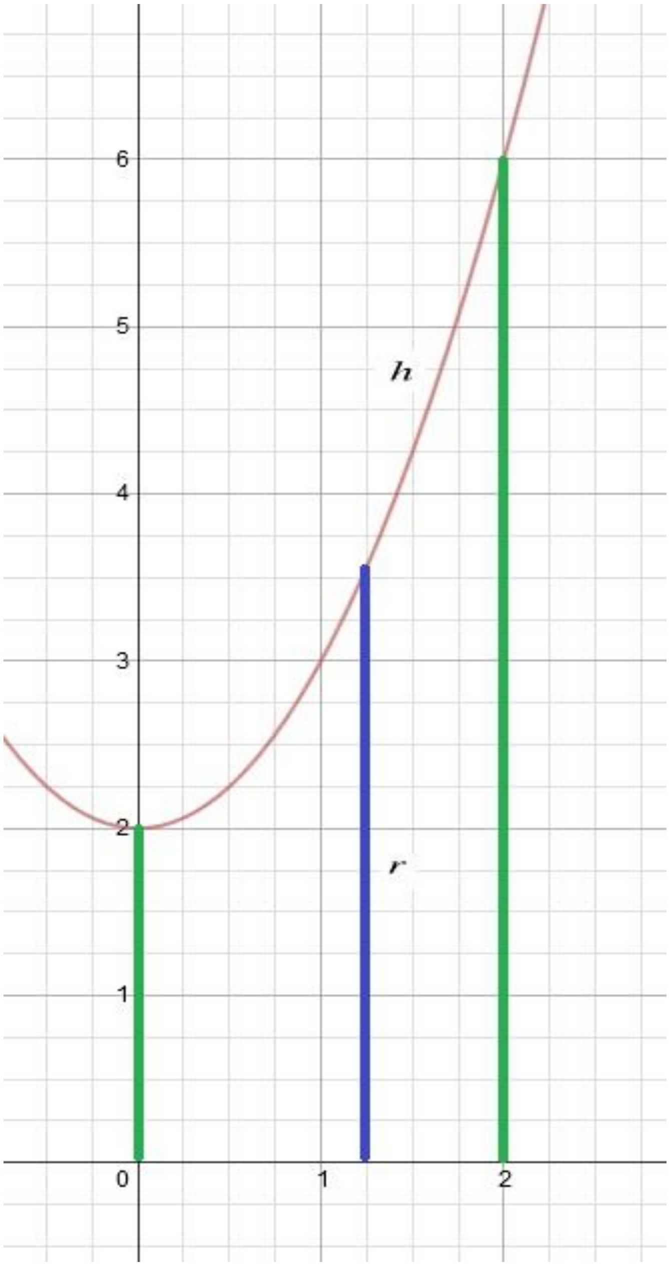
$$S=\int_a^b 2\pi \underbrace{f(x)}_{\text{Radius}}\underbrace{\sqrt{1+\left[f'(x)\right]^2}}_{\text{Arc Length (height)}}\,dx$$

To rotate a function around the Y-axis, the formula is very similar. Where X=h(y). And our limits are our start and end points on the Y-axis

$$S=\int_c^d 2\pi \underbrace{h(y)}_{\text{Radius}}\underbrace{\sqrt{1+\left[h'(y)\right]^2}}_{\text{Arc Length (height)}}\,dy$$

Now for a simple example to expand on this idea.

The function $y = x^2 + 3$ rotated about the X-axis. From 0 to 2



We'll break this up into parts after looking at our graph.

We can see that the green lines are the ends of our cylinder, our "h" is our arc length, and the blue line is the radius at an arbitrary point.

Now remember our formula from earlier, and lets figure out each part

$$S = \int_a^b 2\pi \underbrace{f(x)}_{\text{Radius}} \underbrace{\sqrt{1+\left[f'(x)\right]^2}}_{\text{Arc Length (height)}}\, dx$$

Our limits of integration, A and B are our start and end, which would be 0 and 2.

Our radius would be the Y value at a given X value, simply put this is our function

$$f(x) = x^2 + 3$$

And last, our arc length is given by the square root of 1 plus the derivative of our function squared.

$$\sqrt{1+\left[f'(x)\right]^2}$$

$$\sqrt{1+(x+0)^2}$$

$$\sqrt{1+x^2}$$

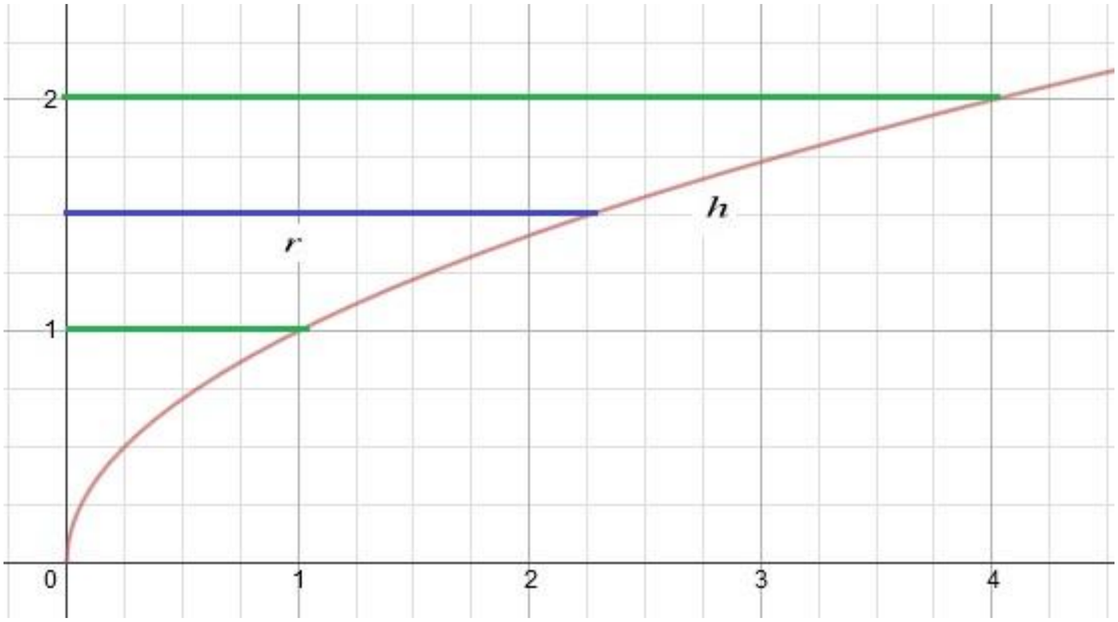
Now we just need to put all of these parts together and arrive at our integral. After solving the integral, you will have the surface area.

$$\int_0^2 2\pi \underbrace{(x^2+3)}_{\text{Radius}} \underbrace{\sqrt{1+x^2}}_{\text{Arc Length (height)}}$$

So now lets see how we could do a similar problem rotated about the Y-axis. We just have to remember the same basic principal of using the radius and arc length in our integral.

Rotate $y = \sqrt{x}$ about the Y axis. From 1 to 2.

Here is our graph with similar markings. The blue line would be our radius at an arbitrary point, and the green lines are our bounds. This problem can be done in two different ways. With using X or Y as our variable. I'll show both ways as I go along. Not all problems can be done both ways, and on some problems, one way proves much simpler than the other.



First, let's figure out our bounds. If we are going to be using X, our bounds are from 1 to 4. If we are going to be using Y, our bounds are from 1 to 2.

Next, lets figure out our radius:
If we are using X, then our radius is the same as our X value, so it's just "X".
Things aren't as simple if we are using Y. We solve our function for X, so we know what we have to do to our "Y" input to get our X value.

So $y = \sqrt{x}$ turns $y^2 = x$ into by squaring both sides.

Now to figure out our "length" by using arc length.

If we are using X, we would have to take the derivative of our original function. And after we have the derivative, we can put it in our arc length formula.

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \qquad \sqrt{1 + (\frac{1}{2\sqrt{x}})^2}$$

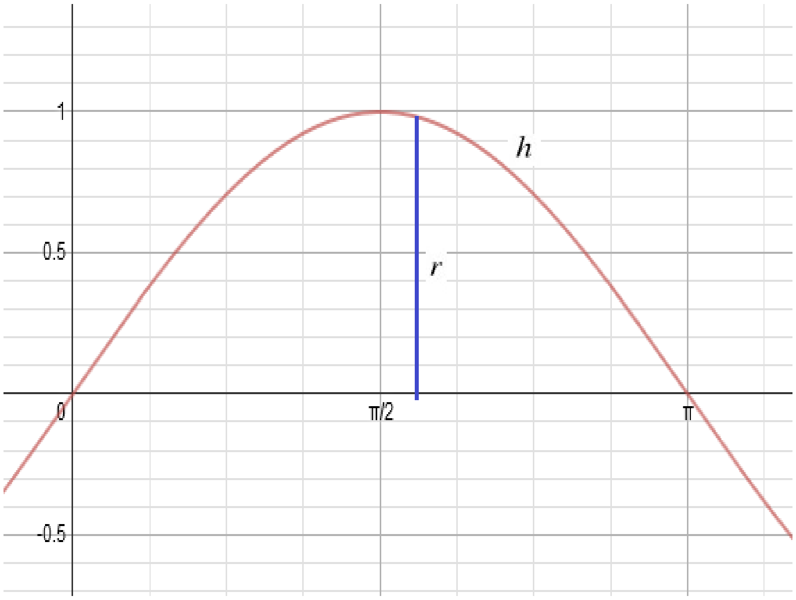
For our Y version, we would just take the derivative of our new function. And plug that into our arc length formula

$$\frac{d}{dy} y^2 = 2y \qquad \sqrt{1+(2y)^2}$$

So now we have all the pieces for both approaches. Put together, here are our two integrals. One in terms of Y, the other in terms of X. Both integrals will give you the same value, but clearly the one on the left is easier to solve.

$$\int_1^2 2\,\pi\, \underbrace{y^2}_{\text{Radius}} \underbrace{\sqrt{1+(2\,y)^2}}_{\text{Arc Length (height)}}\; d\,y \qquad \int_1^4 2\,\pi \underbrace{(x)}_{\text{Radius}} \sqrt{\underbrace{1+\left(\frac{1}{2\sqrt{x}}\right)^2}_{\text{Arc Length (height)}}}\; d\,x$$

Now lets look at $y = \sin x$ rotated around the X axis from 0 to PI



First, our bounds are 0 and PI

Next, let's look at our radius, or our Y value, which is just given by the function. $y = \sin x$

The last thing we need to figure out is our "h" by using our arc length formula. $\sqrt{1+[f'(x)]^2}$

$$\frac{d}{dx}\sin x = \cos x$$

$$\sqrt{1+(\cos x)^2}$$

Now we can put it all together and get our integral. After solving, we will have our surface area.

$$\int_0^{\pi} (2\,\pi)\, \underbrace{\sin(x)}_{\text{Radius}} \underbrace{\sqrt{1+\cos^2(x)}}_{\text{Arc Length (height)}}\; d\,x$$